

# POSITIVITY OF LOG CANONICAL DIVISORS AND MORI/BRODY HYPERBOLICITY

STEVEN S. Y. LU AND DE-QI ZHANG

**ABSTRACT.** Let  $X$  be a complex projective variety of dimension  $n$ ,  $D$  a reduced divisor with a decomposition  $D = \sum_{i=1}^r D_i$ , where the  $D_i$ 's are reduced Cartier but not necessarily irreducible. The pair  $(X, D)$  is called Brody hyperbolic, respectively Mori hyperbolic, with respect to the decomposition if neither  $X \setminus D$  nor  $(\cap_{i \in I} D_i) \setminus (\cup_{j \in J} D_j)$  contains a non constant holomorphic image, respectively algebraic image, of  $\mathbb{C}$  for every partition of  $\{1, \dots, r\} = I \amalg J$ . Assuming that the singularities of the pair  $(X, D)$  are mildly singular, we show that the log canonical divisor  $K_X + D$  is numerically effective in the case of Mori hyperbolicity and that  $K_X + D$  is ample provided that either  $n < 4$  and  $D$  is non-empty or at least  $n - 2$  of the  $D_i$ 's are ample in the case of Brody hyperbolicity.

## 1. INTRODUCTION

Various notions of hyperbolicity have played important guiding roles in geometry and analysis throughout the centuries. In complex geometry, an intrinsic notion via the non-degeneracy of a holomorphically invariant pseudo-distance was formulated by S. Kobayashi from which arose one of his first problems, which demands that the canonical bundle of a compact complex manifold be ample provided that the manifold is hyperbolic, see [6]. This problem remains open in dimension two and, in the projective category, in dimension three and above. In the compact complex category, a well known criterion of Brody characterizes hyperbolicity by the absence of non-constant holomorphic images of  $\mathbb{C}$  (Brody hyperbolicity) and is the most natural notion in our context, see [9, Chap. 3]. From this, an intimately related problem comes to the fore, which demands the numerical effectiveness (nefness) of the canonical bundle of a complex projective manifold in the absence of rational curves. This problem was completely solved by Mori's bend and break theorem. We are motivated by these same problems generalized to the category of singular projective pairs.

We now provide a quick complex hyperbolic geometric perspective to our problem, and to its formulation, via the conjecturally equivalent notions (again by S. Kobayashi, see [9, Chap. 4] and [6, Chap. 9]) of measure hyperbolicity and volume hyperbolicity

---

2000 *Mathematics Subject Classification.* 32Q45, 14E30.

*Key words and phrases.* Brody hyperbolic, ample log canonical (or adjoint) divisor.

with some modern ingredients thrown in. Without going into their definitions, the key point is that, for a complex space, it is measure hyperbolic if it is hyperbolic while it is projective and volume hyperbolic only if it is of general type (op. cit.). Thus modulo the (known) facts about the canonical model of a projective variety of general type, whose existence we assume, the equivalence of these notions would give an affirmation of our problem above. In the case of a pair  $(X, D)$ , where  $X$  is projective and  $D$  is a divisor on  $X$ , the natural notion of hyperbolicity would be that of hyperbolic embedding of  $X \setminus D$  in  $X$ . The condition implies that  $X \setminus D$  is hyperbolic. When  $D$  is a sum of Cartier divisors  $D = \sum_{i=1}^r D_i$ , Brody criterion says that this hyperbolic embedding is guaranteed by (what we call) stratified hyperbolicity: that is the Brody hyperbolicity of  $X \setminus D$  and that of  $(\cap_{i \in I} D_i) \setminus (\cup_{j \in J} D_j)$  for every partition of  $\{1, \dots, r\} = I \amalg J$ . Thus modulo the equivalence of measure hyperbolicity and being of log general type for  $X \setminus D$ , and in a setting such that  $K_X + D$  can be used to define  $X \setminus D$  to be of log general type, stratified hyperbolicity of  $(X, D)$  would imply the ampleness of  $K_X + D$  by the same token.

**Results:** We work over the field  $\mathbb{C}$  of complex numbers. A Zariski closed subset  $X$  in an algebraic or analytic variety is *Brody hyperbolic* = BH (resp. algebraic Brody hyperbolic = ABH, Mori hyperbolic = MH) if the following hypothesis (BH) (resp. (ABH), (MH)) is satisfied:

- (BH) Every holomorphic map from the complex line  $\mathbb{C}$  to  $X$  is a constant map.
- (ABH) No curve in  $X$  has normalization equal to an elliptic curve,  $\mathbb{P}^1$ ,  $\mathbb{C}$  or  $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ .
- (MH) No curve in  $X$  has normalization equal to  $\mathbb{P}^1$  or  $\mathbb{C}$ .

Clearly, (BH)  $\Rightarrow$  (ABH)  $\Rightarrow$  (MH).

Consider a pair  $(X, D)$  of a variety  $X$  and a reduced Weil divisor  $D$  on  $X$ . The pair is called *projective* if  $X$  is a projective variety and *smooth projective* if further  $X$  is smooth and  $D$  is of simple normal crossings.

When  $X$  is an irreducible curve and  $D$  a finite subset of  $X$ , the pair  $(X, D)$  is *Brody hyperbolic* (resp. ABH, MH) if  $X \setminus D$  is Brody hyperbolic (resp. ABH, MH). Inductively, for an irreducible normal variety  $X$  and a reduced Weil divisor  $D$  of  $X$ , the pair  $(X, D)$  is *Brody hyperbolic* (resp. ABH, MH) if  $X \setminus D$  and all pairs  $(D_k, (D - D_k)|_{D_k})$ , with  $D_k$  an irreducible component of  $D$ , are so.

Here, it is quite natural to assume that all the  $D_k$ 's are Cartier or at least  $\mathbb{Q}$ -Cartier from the perspective of hyperbolic geometry.

The pair  $(X, D)$  is called *BH, ABH or MH with respect to a (Cartier) decomposition* of  $D = \sum_{i=1}^r D_i$ , if each  $D_i$  is a reduced (Cartier) divisor, not necessarily irreducible, and both  $X \setminus D$  and  $(\cap_{i \in I} D_i) \setminus (\cup_{j \in J} D_j)$  are respectively BH, ABH or MH for every partition of  $\{1, \dots, r\} = I \amalg J$ . If this is so with respect to a Cartier decomposition, we will call

the respective notions of hyperbolicity *stratified*. Note that a Cartier decomposition of  $D$  implies that  $D$  is Cartier and that the respective hyperbolicities for a smooth pair  $(X, D)$  are equivalent to the same with respect to the irreducible Cartier decomposition of  $D$ .

*In this paper, by hyperbolic, we mean Brody hyperbolic.*

We consider the following conjecture:

**Conjecture 1.1.** *Let  $(X, D)$  be a smooth projective Mori hyperbolic (resp. Brody hyperbolic) pair. Then the log canonical divisor  $K_X + D$  is numerically effective (resp. ample).*

Our main theorems below give an affirmative answer to Conjecture 1.1 generalized to singular pairs<sup>1</sup> assuming further conditions for the ampleness of  $K_X + D$ , such as the ampleness of at least  $n - 2$  ( $n := \dim X$ ) irreducible Cartier components of  $D$ .

**Theorem 1.2.** *Let  $(X, D)$  be a projective Brody hyperbolic pair with  $n := \dim X$ , and  $\Gamma$  an effective Weil  $\mathbb{Q}$ -divisor on  $X$  such that the pair  $(X, D + \Gamma)$  is divisorial log terminal (dlt). Assume one of the following three conditions.*

- (1)  $n \leq 2$ .
- (2)  $n = 3$ ;  $D$  is non-empty and Cartier.
- (3)  $(X, D)$  is a smooth projective pair;  $D$  has at least  $n - 2$  irreducible components amongst which at least  $n - 3$  are ample.

*Then  $K_X + D + \Gamma$  is ample.*

One may take  $\Gamma = 0$  in all Theorems 1.2  $\sim$  1.4. However, in the inductive procedure of our proof, such a  $\Gamma$  naturally appears as the ‘different’ in the adjunction formula (see [1, Proposition 3.9.2], or Lemma 2.2 below).

**Theorem 1.3.** *Let  $(X, D)$  be a projective Mori hyperbolic pair with respect to a Cartier decomposition  $D = \sum_{i=1}^r D_i$  for some  $r \geq 0$ . Let  $\Gamma$  be an effective Weil  $\mathbb{Q}$ -divisor on  $X$  such that the pair  $(X, D + \Gamma)$  is divisorial log terminal (dlt). Then  $K_X + D + \Gamma$  is numerically effective.*

*In particular,  $K_X + D$  is numerically effective for a smooth Mori hyperbolic pair  $(X, D)$ .*

By virtue of Theorem 1.3, a positive answer to the abundance conjecture (not known in dimension  $\geq 4$ ) would imply, by induction on the dimension, that the  $K_X + D$  in Conjecture 1.1 in the case of Mori hyperbolicity is either a nef and big divisor, or trivial.

---

<sup>1</sup>Here by a singular pair, we will assume that it is a dlt pair (see §2 below). The assumption is natural (and in many respect the most general) as our proof is by induction on dimension from running the LMMP for singular pairs. It implies an explicit adjunction formula (Lemma 2.2) and that  $k$ -fold intersections of components of  $D$  are of pure codimension- $k$  in  $X$ , crucial in our inductive procedure.

**Theorem 1.4.** *Let  $(X, D)$  be a projective Brody hyperbolic pair with respect to a Cartier decomposition  $D = \sum_{i=1}^r D_i$  for some  $r \geq n - 2$ ,  $n := \dim X$ . Let  $\Gamma$  be an effective Weil  $\mathbb{Q}$ -divisor on  $X$  such that the pair  $(X, D + \Gamma)$  is divisorial log terminal (dlt). Assume one of the following two conditions:*

- (1)  $n = 4$ ;  $r \geq 2$ ;  $D_1$  is irreducible and ample.
- (2)  $n \geq 4$ ;  $r \geq n - 2$ ;  $D_j$  is ample for all  $j \leq n - 2$ .

*Then  $K_X + D + \Gamma$  is ample.*

**Remark 1.5.** (1) The proofs of the above theorems are independent of the results of Keel-McKernan [5] and of Miyanishi-Tsunoda [10], which are only used in the proof of Proposition 3.6.

(2) If we remove the condition  $D \neq \emptyset$  in Theorem 1.2(2) then the proof shows that either  $K_X$  is ample or  $K_X \sim_{\mathbb{Q}} 0$ . In the latter case, if  $X$  is further assumed to be smooth, the Bogomolov decomposition and the fact that a finite cover of  $X$  is still hyperbolic imply that there is a finite étale cover  $X' \rightarrow X$  such that  $X'$  is a Calabi-Yau manifold of dimension three in the narrow sense:  $X'$  is simply connected and  $K_{X'} \sim 0$ .

(3) It is a long standing question whether there exist rational or elliptic curves on a Calabi-Yau manifold, known to exist in dimension two [11], but unknown in dimension three or higher. If exist in dimension three, it would allow us to remove the condition  $D \neq \emptyset$  in Theorem 1.2(2), at least for smooth threefolds, and also weaken the conditions in Theorem 1.2(3).

(4) Our search for a natural setting for the problems in the singular setting seems to lead us to more questions than answers, even for surfaces. For example, from the proof of Proposition 2.3 arise the spectre of the existence of Mori hyperbolic dlt surface pair whose log canonical divisor is not numerically effective. The inductive nature of our proof shows that many of these questions have their roots in dimension two. Our results appears to be new even for smooth surface pairs.

Concerning our proof in the presence of Cartier boundary divisors, a key role is played by Kawamata's result on the length of extremal rays, see Lemma 3.1. Our proofs of Theorems 1.2 ~ 1.4 are inductive in nature and reduce the problem to weaker questions about the pseudo-effectivity or the nefness of adjoint divisors of subvarieties of  $X$ . The log minimal model program (LMMP) is run formally without going into its technical details and the established abundance theorem in dimension  $\leq 3$  is also used.

**Acknowledgement:** The first author would like to acknowledge his support from an NSERC discovery grant. This paper is done during the visits of the second author

to UQAM in August 2011 and July 2012, with the support and the excellent research environment provided by CIRGET there; he is also supported by an ARF of NUS.

## 2. PRELIMINARIES

2.1. The basics of the Minimal Model Program (MMP) as found in [8], whose notation we adopt, will be used freely and often implicitly. By a divisor on a normal variety  $X$ , we will always mean a Weyl  $\mathbb{Q}$ -divisor. However, a Cartier divisor will remain integral. Let  $L$  be a  $\mathbb{Q}$ -Cartier divisor on a projective variety  $X$ .  $L$  is *pseudo-effective* if its class in the Neron-Severi space  $\mathrm{NS}_{\mathbb{R}}(X) := \mathrm{NS}(X) \otimes \mathbb{R}$  is in the closure of the cone generated by the classes of effective divisors in  $\mathrm{NS}_{\mathbb{R}}(X)$ .  $L$  is *numerically effective* (or *nef*) if  $\deg(L|_C) \geq 0$  for every curve  $C$  on  $X$ .

Let  $\Gamma = \sum_{i=1}^r a_i \Gamma_i$  be a divisor on  $X$  with  $\Gamma_i$  distinct irreducible divisors and  $a_i \in \mathbb{Q}$ . Its *integral part* is given by  $\lfloor \Gamma \rfloor := \sum_{i=1}^r \lfloor a_i \rfloor \Gamma_i$  where  $\lfloor a_i \rfloor$  is the integral part of the rational number  $a_i$ . We say that  $\Gamma$  is *fractional* if  $a_i \in (0, 1)$  for all  $i$ .

Recall that a pair  $(X, \Delta)$  of a divisor  $\Delta$  on a normal variety  $X$  is called *divisorial log terminal* = *dlt* (resp. *Kawamata log terminal* = *klt*, or *log canonical* = *lc*) if for some log resolution (resp. for one (or equivalently for all) log resolution), the discrepancy of every exceptional divisor (resp. of every divisor) on the resolution satisfies  $> -1$  (resp.  $> -1$ , or  $\geq -1$ ). In all cases,  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier. When  $(X, \Delta)$  is dlt, the integral part  $\lfloor \Delta \rfloor$  is normal crossing in codimension-one. A dlt pair  $(X, \Delta)$  is klt if and only if  $\lfloor \Delta \rfloor = 0$ . In case  $\dim X = 2$ , if  $(X, \Delta)$  is dlt, then  $X$  is  $\mathbb{Q}$ -factorial (so that all Weyl divisors are  $\mathbb{Q}$ -Cartier). We refer to [8, Definition 2.34] and [2, §3] for details.

The following adjunction formula as given, for example, by [1, Proposition 3.9.2] is crucial in our induction process to reduce the problem to lower dimensions.

**Lemma 2.2.** *Let  $X$  be a normal variety,  $D$  a reduced Weil divisor and  $\Gamma \geq 0$ . Suppose that  $(X, D + \Gamma)$  is dlt and  $D_1 \subseteq D$  is an irreducible component. Then  $D_1$  is a normal variety and we can write*

$$(K_X + D + \Gamma)|_{D_1} = K_{D_1} + (D - D_1)|_{D_1} + \Delta$$

*such that  $\Delta$  is an effective Weil divisor,  $(D - D_1)|_{D_1}$  is a reduced Weil divisor and the pair  $(D_1, (D - D_1)|_{D_1} + \Delta)$  is dlt.*

The following result yield the start of one of our key induction processes.

**Proposition 2.3.** *Let  $(X, D)$  be a projective Brody hyperbolic pair with  $n := \dim X \leq 2$  and  $\Gamma$  an effective Weil  $\mathbb{Q}$ -divisor such that the pair  $(X, D + \Gamma)$  is dlt. Then  $K_X + D + \Gamma$  is an ample divisor.*

*Proof.* When  $\dim X = 1$ , it is clear. Assume that  $\dim X = 2$ . Since  $(X, D + \Gamma)$  is dlt,  $X$  is  $\mathbb{Q}$ -factorial (see [8, Proposition 4.11]).

Assume that  $K_X + D + \Gamma$  is not nef. By the cone theorem [8, Theorem 3.4.7], there is an extremal rational curve and let  $\xi : X \rightarrow Z$  be the corresponding contraction with  $F$  a general fibre. Then  $\xi$  is either a Fano or divisorial contraction.

Consider the case that  $\xi$  is a Fano contraction. Then  $-(K_X + D + \Gamma)|_F$  is ample. Using this fact, we will reach a contradiction. Indeed, if  $\dim Z \geq 1$  (resp.  $\dim Z = 0$  and  $D \neq 0$ ), we get a contradiction to the ampleness of  $(K_X + D)|_F = K_F + D|_F$  ( resp. of  $(K_X + D + \Gamma)|_G = K_G + (D - G)|_G + \Delta$  with  $\Delta \geq 0$ , and  $G$  an irreducible component of  $D$  ) by the adjunction formula of Lemma 2.2 and since the proposition is true in dimension one. When  $\dim Z = 0$  and  $D = 0$ , the ampleness of  $-(K_X + \Gamma)$  implies that  $X$  is uniruled, contradicting the hyperbolicity of  $X = X \setminus D$ . Hence  $\xi : X \rightarrow Z$  is not a Fano contraction.

Consider the case that  $\xi$  is a divisorial (i.e., birational) contraction. Since  $(X, D)$  is lc, so is  $(Z, \xi_* D)$ . If the rational curve  $\ell$  is contained in a component of  $D$ , written  $\ell \subseteq D$  by abuse of notation, it would be normal and meet the other components of  $D$  at two or less points; if  $\ell \not\subseteq D$ , then either  $\ell$  is nodal and disjoint from  $D$ , or  $\ell$  is a smooth rational curve and would meet  $D$  at two or less points (otherwise, the image  $\xi(\ell)$  would be a point on  $Z$  through which there are at least three local irreducible components of  $\xi_* D$ , contradicting the log canonicity of the pair  $(Z, \xi_* D)$ ; see [8, Theorem 4.7], or [7, page 57-58]). This contradicts the hyperbolicity of  $(X, D)$ .

Therefore,  $K_X + D + \Gamma$  is nef. By the abundance theorem in dimension  $\leq 3$  (see [8, §3.13]), there is a morphism  $\sigma : X \rightarrow Y$  with a connected general fibre  $F$ , and an ample divisor  $H$  on  $Y$  such that  $K_X + D + \Gamma = \sigma^* H$ . Suppose the contrary that  $K_X + D + \Gamma$  is not ample (and we will get a contradiction). If  $\dim Y = 1$ , then  $(K_X + D + \Gamma)|_F \sim_{\mathbb{Q}} 0$ , contradicting the hyperbolicity of  $(F, D|_F)$ , since our proposition is true in dimension one.

Consider the case that  $\dim Y = 2$ , i.e.,  $\sigma$  is birational and contracts at least one curve  $\ell$ . As  $K_X + D + \Gamma = \sigma^*(K_Y + \sigma_*(D + \Gamma))$ , the pair  $(Y, \sigma_*(D + \Gamma))$  is still lc and so  $\ell$  is either a rational curve or an arithmetic-genus one curve. If  $\ell$  is a rational curve then we argue as in the non-nef case above to give a contradiction to hyperbolicity. If it is an arithmetic genus one curve, then the log canonicity implies that either it is a connected component of  $D$  or it is disjoint from  $D$  (see [8, Theorem 4.7], or [7, page 57-58]). This contradicts the hyperbolicity of  $(X, D)$ .

We still have to consider the case  $\dim Y = 0$ , i.e.,  $K_X + D + \Gamma \sim_{\mathbb{Q}} 0$ . If  $G$  is an irreducible component of  $D$ , then  $(K_X + D + \Gamma)|_G \sim_{\mathbb{Q}} 0$  gives a contradiction to the ampleness result of the curve case, as in the Fano contraction case above.

Thus, we may assume that  $D = 0$  and  $K_X + \Gamma \sim_{\mathbb{Q}} 0$ . If  $\Gamma \neq 0$ , then  $X$  is uniruled, contradicting the hyperbolicity of  $X = X \setminus D$ . Suppose that  $\Gamma = 0$  so that  $K_X \sim_{\mathbb{Q}} 0$ . Let  $X' \rightarrow X$  be the global index-one cover (unramified in codimension-one) so that  $K_{X'} \sim 0$  and  $X'$  has at worst canonical singularities. Hence the minimal resolution of  $X'$  satisfies  $K_{X'} \sim_{\mathbb{Q}} 0$ . By the classification of surfaces and noting that a K3 surface has infinitely many singular elliptic curves (see [11]), we see that  $X$  is non-hyperbolic. This is a contradiction. Therefore, we have proved that  $K_X + D + \Gamma$  is ample. This proves Proposition 2.3.  $\square$

**Lemma 2.4.** *Let  $n \geq 2$ . Assume that Theorems 1.2  $\sim$  1.4 are true under the condition of 1.2(3) (resp. 1.3, or 1.4(2)) for all pairs of dimension  $n$ . Let  $(X, D + \Gamma)$  be a pair as in Theorem 1.2(3) (resp. 1.3, or 1.4(2)) but with  $\dim X = n + 1$ . Let  $Y$  be an irreducible component of  $D + [\Gamma]$ . Then  $(K_X + D + \Gamma)|_Y$  is a nef (resp. ample) divisor on  $Y$  in the case of Theorem 1.3 (resp. in the case of Theorem 1.2(3) and 1.4(2)).*

*Proof.* By the adjunction formula of Lemma 2.2,

$$(K_X + D + \Gamma)|_Y = K_Y + (D + [\Gamma] - Y)|_Y + \Delta,$$

where  $\Delta \geq 0$  and the pair  $(Y, D_Y + \Gamma_Y)$  is dlt. Here if the hyperbolicity is with respect to a Cartier decomposition  $D = \sum_{i=1}^r D_i$ , we define  $D_Y := (\sum_{i \neq k} D_i)|_Y$  and  $\Gamma_Y := (D_k - Y + [\Gamma])|_Y + \Delta$  when  $Y \subseteq D_k$  (resp.  $D_Y := D|_Y$  and  $\Gamma_Y := ([\Gamma] - Y)|_Y + \Delta$  when  $Y \subseteq [\Gamma]$ ). Otherwise, we define  $D_Y := (D - Y)|_Y$  and  $\Gamma_Y := [\Gamma]|_Y + \Delta$  when  $Y \subseteq D_k$  (resp.  $D_Y := D|_Y$  and  $\Gamma_Y := ([\Gamma] - Y)|_Y + \Delta$  when  $Y \subseteq [\Gamma]$ ). Note that the pair  $(Y, D_Y)$  still satisfies the respective conditions in Theorems 1.2(3), 1.3 and 1.4(2), by an easy check. Now the result follows from the assumption.  $\square$

The result below follows from Proposition 2.3 and the proof of Lemma 2.4.

**Corollary 2.5.** *Let  $(X, D)$  be a projective Brody hyperbolic pair with  $n := \dim X \leq 3$  and  $\Gamma \geq 0$  a Weil  $\mathbb{Q}$ -divisor such that the pair  $(X, D + \Gamma)$  is dlt. Then the restriction  $(K_X + D + \Gamma)|_G$  is an ample divisor on  $G$  for every irreducible component  $G$  of  $D + [\Gamma]$ .*

**Remark 2.6.** From the proof of Proposition 2.3, if we weaken BH to ABH in the assumption, then either  $K_X + D + \Gamma$  is ample, or  $D = \Gamma = 0$  and  $X$  is a klt surface with  $K_X \sim_{\mathbb{Q}} 0$  which implies that  $X$  is a simple abelian surface since  $X$  is ABH.

### 3. PROOFS OF THEOREMS

In this section, we prove Theorems 1.2  $\sim$  1.4.

Kawamata's result below is very powerful for our proof.

**Lemma 3.1.** (see [4, Theorem 1]) *Let  $(X, \Delta)$  be a dlt pair and  $g : X \rightarrow Y$  the contraction of a  $(K_X + \Delta)$ -negative extremal ray  $R = \mathbb{R}_{>0}[\ell]$ . Let  $E$  be an irreducible component of the  $g$ -exceptional locus, i.e., the set of points along which  $g$  is not isomorphism. Set  $d := \dim E - \dim g(E)$ . Then  $E$  is covered by a family of rational curves  $\{\ell_t\}$  such that  $g(\ell_t)$  is a point and  $-\ell_t.(K_X + \Delta) \leq 2d$ . The equality holds only when  $(X, \Delta)$  is non-klt or  $g$  is a Fano contraction.*

*Proof.* If  $(X, \Delta)$  is klt, then the lemma is just [4, Theorem 1]. When  $(X, \Delta)$  is dlt, by [8, Proposition 2.43], for any ample divisor  $H$  on  $X$ , there is a constant  $c > 0$  (depending on  $H$ ) such that for every  $0 < \varepsilon \ll 1$ , one can find a divisor  $\Delta'$  on  $X$  with  $\Delta' \sim_{\mathbb{Q}} \Delta + \varepsilon cH$  and  $(X, \Delta')$  klt. We choose  $\varepsilon$  small enough such that  $\ell$  is still  $(K_X + \Delta')$ -negative. Now [4] applies and  $E$  is covered by  $g$ -contractible rational curves  $\ell_t$  with  $-\ell_t.(K_X + \Delta') \leq 2d$ . Let  $\varepsilon \rightarrow 0$ . The lemma follows.  $\square$

### 3.2. Proof of Theorem 1.3

Let  $(X, D + \Gamma)$  be as in Theorem 1.3, so  $(X, D)$  is Mori hyperbolic with respect to a Cartier decomposition  $D = \sum_{i=1}^r D_i$  for some  $r \geq 0$ . Let  $n := \dim X$ . The case  $n = 1$  is clear. Thus we may assume that  $n \geq 2$ . We proceed by induction on  $n$ . We apply the LMMP to  $(X, D + \Gamma)$  (see [2, Theorem 1.1]).

Suppose on the contrary that  $K_X + D + \Gamma$  is not nef. Then there is a  $(K_X + D + \Gamma)$ -negative extremal ray  $R = \mathbb{R}_{>0}[\ell]$  with  $\ell$  a rational curve. Let  $f : X \rightarrow X_1$  be the corresponding extremal contraction, and  $\text{Exc}(f) \subset X$  the  $f$ -exceptional locus, i.e., the set of points along which  $f$  is not an isomorphism. Let  $[\Gamma]$  be the integral part of  $\Gamma$ . If  $\ell \subset G$ , an irreducible component of  $D + [\Gamma]$ , then  $0 > \ell.(K_X + D + \Gamma) = \ell.(K_X + D + \Gamma)|_G$ . This contradicts the nefness result in lower dimension by the inductive assumption (see the proofs of Lemma 2.4 and Corollary 2.5).

Therefore, we may assume that  $\ell \not\subset (D + [\Gamma])$  for any  $[\ell] \in R$ , and hence  $E_i \not\subset (D + [\Gamma])$  for every irreducible component  $E_i$  of  $\text{Exc}(f)$ ; this always holds if  $f$  is a Fano contraction. Since  $\ell \not\subset D$ ,  $0 > \ell.(K_X + D + \Gamma) \geq \ell.(K_X + \Gamma)$ . Hence  $R$  is also a  $(K_X + \Gamma)$ -negative extremal ray, and  $(X, \Gamma)$  is dlt, see [8, Corollary 2.39] (to be used later).

Since  $D$  is Cartier,  $D|_{E_1}$  is an effective Cartier divisor on  $E_1$ . If  $D = 0$  or if  $D \cap E_1 = \emptyset$ , then  $E_1 = E_1 \setminus D$  is Mori hyperbolic. This contradicts Lemma 3.1. If  $f|_{E_1 \cap D}$  contracts a curve, also denoted as  $\ell$ , to a point on  $X_1$ , then  $[\ell] \in R$  and  $\ell \subset D$ , which is a contradiction as we have just seen. Thus,  $f|_{E_1 \cap D}$  is a finite morphism. Hence

$$\dim f(E_1) \geq \dim f(E_1 \cap D) = \dim E_1 \cap D = \dim E_1 - 1$$

so  $\dim f(E_1) = \dim E_1 - 1$ .



By Lemma 3.1,  $E_1$  is covered by  $f$ -contractible rational curves  $\ell$  such that

$$-\ell.(K_X + \Gamma) \leq 2(\dim E_1 - \dim f(E_1)) = 2.$$

Thus  $\ell.(K_X + \Gamma) \geq -2$ . Now  $0 > \ell.(K_X + D + \Gamma) \geq -2 + \ell.D$ . Hence, if  $\nu : \tilde{\ell} \rightarrow \ell$  is the normalization then  $2 > \ell.D = \nu^*(D|_{\ell})$ . So,  $D$  being Cartier and integral, the normalization of  $\ell \setminus D$  contains  $\mathbb{C}$ . This contradicts the Mori hyperbolicity of  $X \setminus D$ . Hence Theorem 1.3 is true.

### 3.3. Proof of Theorems 1.2 and 1.4

We proceed by induction on  $n := \dim X$ . If  $n \leq 2$ , the ampleness follows from Proposition 2.3. Thus we may assume that  $n \geq 3$ . The same argument as in 3.2 proves the nefness of  $K_X + D + \Gamma$  for both Theorems 1.2 and 1.4.

**Lemma 3.4.**  *$K_X + D + \Gamma$  is ample when  $D_1$  is ample and Cartier (this holds when  $n \geq 4$ ).*

*Proof.* Suppose on the contrary that  $K_X + D + \Gamma$  is not ample. Then, by Kleiman's ampleness criterion, there is an effective 1-cycle  $[\ell]$  such that  $\ell.(K_X + D + \Gamma) = 0$ . Since there is an ample  $D_1 \subset D$ , we can write  $D = D_\varepsilon + \Delta_\varepsilon$  with an ample  $\mathbb{Q}$ -Cartier divisor  $\Delta_\varepsilon = \varepsilon_1 D_1 + \varepsilon_2 D$  for some  $\varepsilon_i \in (0, 1)$ . Now  $(X, D_\varepsilon + \Gamma)$  is still dlt (see [8, Corollary 2.39]). Note that

$$0 = \ell.(K_X + D + \Gamma) = \ell.(K_X + D_\varepsilon + \Gamma) + \ell.\Delta_\varepsilon > \ell.(K_X + D_\varepsilon + \Gamma).$$

By the cone theorem [2, Theorem 1.1],  $\ell$  is parallel to  $\ell' + \ell''$  for some effective 1-cycle  $\ell'$  and a  $(K_X + D_\varepsilon + \Gamma)$ -negative extremal rational curve  $\ell''$ . Note that the nef divisor  $K_X + D + \Gamma$  is perpendicular to  $\ell$  and hence

$$0 = \ell''.(K_X + D + \Gamma) = \ell'.(K_X + D + \Gamma).$$

Let  $g : X \rightarrow X_2$  be the extremal contraction corresponding to the  $(K_X + D_\varepsilon + \Gamma)$ -negative extremal ray  $\mathbb{R}_{>0}[\ell'']$ . If  $\ell''$  lies in an irreducible component  $G$  of  $D + [\Gamma]$ , then

$$0 = \ell''.(K_X + D + \Gamma) = \ell'.(K_X + D + \Gamma)|_G$$

contradicting the ampleness result in lower dimension by the inductive assumption (see the proofs of Lemma 2.4 and Corollary 2.5). So we may assume that  $\ell'' \not\subset (D + [\Gamma])$ . Hence  $\ell'' \cap (D + [\Gamma])$  is a finite set. Thus, no irreducible component  $E_1$  of  $\text{Exc}(g)$  is contained in  $D + [\Gamma]$ . Since  $X \setminus D$  is hyperbolic,  $\ell'' \cap D$  is a non-empty finite set.

Thus  $0 = \ell''.(K_X + D + \Gamma) > \ell''.(K_X + \Gamma)$ . Hence  $\mathbb{R}_{>0}[\ell'']$  is also a  $(K_X + \Gamma)$ -negative extremal ray. By the argument in 3.2 and noting that  $\ell'' \not\subset D$ ,

$$\dim g(E_1) \geq \dim g(E_1 \cap D) = \dim E_1 \cap D = \dim E_1 - 1.$$

By Lemma 3.1,  $E_1$  is covered by rational curves  $\ell''$  with  $-\ell''.(K_X + \Gamma) \leq 2(\dim E_1 - \dim g(E_1)) \leq 2$ . Now  $0 = \ell''.(K_X + D + \Gamma) \geq -2 + \ell''.D$ . Thus  $\ell''.D \leq 2$ , and we reach a contradiction to the Brody hyperbolicity of  $X \setminus D$  as in 3.2. Therefore,  $K_X + D + \Gamma$  is ample, and the lemma is proved.  $\square$

By Lemma 3.4 and the assumption of Theorems 1.2 and 1.4, we may assume that  $n = 3$ ,  $D \neq \emptyset$  and  $D$  is Cartier. Since we have proved the nefness of  $K_X + D + \Gamma$ , by the abundance theorem for log canonical pairs of dimension  $\leq 3$  (see [8, 3.13]), there is a fibration  $h : X \rightarrow Z$  with connected fibres such that

$$K_X + D + \Gamma = h^*H$$

for some ample  $\mathbb{Q}$ -divisor  $H$  on  $Z$ . Let  $F$  be a general fibre of  $h$ .

Case(I)  $\dim Z = 0$ . Then  $K_X + D + \Gamma \sim_{\mathbb{Q}} 0$ . Thus for an irreducible component  $G$  of  $D$ , the divisor  $(K_X + D + \Gamma)|_G$  is trivial, contradicting its ampleness by Corollary 2.5. *This is the place we need  $D \neq \emptyset$ ; see Remark 1.5.*

Case(II).  $0 < \dim Z < \dim X$ . We get a contradiction to the hyperbolicity of the pair  $(F, D|_F)$ . Indeed, note that the adjoint divisor below is numerically trivial

$$K_F + (D + \Gamma)|_F = (K_X + D + \Gamma)|_F = h^*H|_F$$

since  $F$  is a fibre of  $h$ , contradicting its ampleness by Proposition 2.3.

Case(III).  $\dim Z = \dim X$ . Then  $K_X + D + \Gamma$  is nef and big. If  $h : X \rightarrow Z$  is an isomorphism, then we are done. Otherwise, let  $\text{Exc}(h) \subset X$  be the exceptional locus of  $h$  and  $\ell \subseteq \text{Exc}(h)$  a curve contracted by  $h$ . Then  $0 = \ell.h^*H = \ell.(K_X + D + \Gamma)$ .

By the argument in Lemma 3.4 (using Corollary 2.5), we have  $\ell \not\subset (D + \lfloor \Gamma \rfloor)$ . Hence  $\ell \cap (D + \lfloor \Gamma \rfloor)$  is a finite set and  $\ell.D \geq 0$ . Thus, no irreducible component of  $\text{Exc}(h)$  is contained in  $D + \lfloor \Gamma \rfloor$ .

Consider the case that  $\ell.(K_X + \Gamma) < 0$  for some  $h$ -contractible curve  $\ell$ . Then  $\ell$  is parallel to  $\ell' + \ell''$  for some effective 1-cycle  $\ell'$  and a  $(K_X + \Gamma)$ -negative extremal rational curve  $\ell''$  by Mori's cone theorem (see [8, Theorem 3.7] or [2, Theorem 1.1]). Note that the nef divisor  $K_X + D + \Gamma$  is perpendicular to  $\ell$  and hence

$$0 = \ell''.(K_X + D + \Gamma) = \ell'.(K_X + D + \Gamma).$$

Thus  $\ell'' \not\subset (D + \lfloor \Gamma \rfloor)$  by the same argument as above for  $\ell$  (for the later use).

Let  $h_1 : X \rightarrow Z_1$  be the contraction of the extremal ray  $\mathbb{R}_{>0}[\ell'']$ . Since  $\ell''.h^*H = 0$  with  $H$  ample, every irreducible component  $E_1$  of the exceptional locus of  $h_1$  is a subset of  $\text{Exc}(h)$  and hence is not contained in  $D + \lfloor \Gamma \rfloor$ . As in the proofs of Lemma 3.4 and

3.2 and using Lemma 3.1,  $E_1$  is covered by rational curves  $\ell''$  with  $-\ell''.(K_X + \Gamma) \leq 2(\dim E_1 - \dim h_1(E_1)) \leq 2$ . Now  $0 = \ell''.(K_X + D + \Gamma) \geq -2 + \ell''.D$ . Thus  $\ell''.D \leq 2$  and we reach a contradiction to the Brody hyperbolicity of  $X \setminus D$  as in 3.2.

Consider the case that  $\ell.(K_X + \Gamma) \geq 0$  for every curve  $\ell$  on  $X$  contracted by  $h$ . Then  $0 = \ell.(K_X + D + \Gamma) \geq \ell.D \geq 0$ . So  $\ell.(K_X + \Gamma) = \ell.D = 0$  and  $\ell \cap D = \emptyset$ . Thus  $\text{Exc}(h) \cap D = \emptyset$ . When  $h : X \rightarrow Z$  is a small contraction,  $K_X + D + \Gamma = h^*(K_Z + h_*(D + \Gamma))$ . This means that a ‘good log resolution’ for the dlt pair  $(X, D + \Gamma)$  as in [8, Theorem 2.44(2)] is also a ‘good log resolution’ for  $(Z, h_*(D + \Gamma))$  and hence the latter is also dlt. Thus, by the solution of Hacon-McKernan to a conjecture of Shokurov ([3, Corollary 1.5]), every fibre of  $h : X \rightarrow Z$  is rationally chain connected. Hence the above  $\ell$  can be chosen to be a rational curve away from  $D$ , contradicting the hyperbolicity of  $X \setminus D$ .

If  $h$  contracts a surface  $S \subseteq \text{Exc}(h)$ , the fact that  $S \cap D = \emptyset$  would imply that  $S = S \setminus D$  is hyperbolic and hence is not a uniruled surface. This is impossible as shown in Lemma 3.5 below.

This proves the ampleness of  $K_X + D + \Gamma$ , modulo Lemma 3.5 below.

**Lemma 3.5.** *It is impossible that a birational morphism  $\varphi : X \rightarrow W$  of threefolds contracts a non-uniruled surface  $S \subset X$  with  $S \not\subseteq (D + \lfloor \Gamma \rfloor)$ , such that  $C.(K_X + D + \Gamma) \leq 0$  for a general curve  $C$  on  $S$  (resp. a general fibre of  $\varphi|_S : S \rightarrow \varphi(S)$ ) when  $\dim \varphi(S)$  equals 0 (resp. 1).*

*Proof.* Take a  $\mathbb{Q}$ -factorization, i.e., a small partial resolution (see [2, Theorem 10.4])  $\sigma_1 : X' \rightarrow X$  such that  $X'$  is  $\mathbb{Q}$ -factorial and

$$K_{X'} + D' + \Gamma' = \sigma_1^*(K_X + D + \Gamma),$$

where  $D' := \sigma'_1 D$  ( $= \sigma_1^* D$ ,  $D$  being  $\mathbb{Q}$ -Cartier) and  $\Gamma' := \sigma'_1 \Gamma$  are proper transforms of  $D$  and  $\Gamma$  respectively. Suppose the contrary that  $h$  contracts the surface  $S \not\subseteq (D + \lfloor \Gamma \rfloor)$ . Set  $S' := \sigma'_1 S$ , which is now  $\mathbb{Q}$ -factorial. Write  $\Gamma' := aS' + \tilde{\Gamma}'$  for some  $a \in [0, 1)$  such that  $S'$  is not a component of  $\tilde{\Gamma}'$ . Take a dlt blowup (see [2, Theorem 10.4])  $\sigma_2 : X'' \rightarrow X'$  and let  $E_{\sigma_2}$  be the reduced exceptional divisor. We have

$$K_{X''} + D'' + \tilde{\Gamma}'' + S'' + E_{\sigma_2} + E'' = \sigma_2^*(K_{X'} + D' + \tilde{\Gamma}' + S')$$

where the pair  $(X'', D'' + \tilde{\Gamma}'' + S'' + E_{\sigma_2})$  is  $\mathbb{Q}$ -factorial dlt with  $D'' := \sigma'_2 D'$ ,  $\tilde{\Gamma}'' := \sigma'_2 \tilde{\Gamma}'$ ,  $S'' := \sigma'_2 S'$ , and  $E'' \geq 0$  a  $\sigma_2$ -exceptional divisor. Let  $C' \subset S'$  and  $C'' \subset S''$  be the strict transforms of the general curve  $C \subset S$  (contracted by  $\varphi$ ). By the adjunction formula (Lemma 2.2),  $(K_{X''} + S'')|_{S''} = K_{S''} + \Delta$  for some  $\Delta \geq 0$ . Since  $S$  is non-uniruled,  $K_{S''}$

is pseudo-effective, hence so are  $(K_{X''} + S'')|_{S''}$  and  $\sigma_2^*(K_{X'} + D' + \tilde{\Gamma}' + S')|_{S''}$ . Thus

$$\begin{aligned} 0 &\leq C'' \cdot \sigma_2^*(K_{X'} + D' + \tilde{\Gamma}' + S') = C' \cdot \sigma_1^*(K_X + D + \Gamma) + (1 - a)C' \cdot S' \\ &= C \cdot (K_X + D + \Gamma) + (1 - a)C' \cdot S' \leq (1 - a)C' \cdot S'. \end{aligned}$$

Since the composition  $X' \rightarrow X \rightarrow Z$  is birational and contracts  $S'$ , one can choose a general  $C'$  such that  $C' \cdot S' < 0$  by the well-known negativity lemma (reducing to the surface case after cutting  $S'$  by a general hypersurface). This contradicts the inequalities displayed above. This completes the proof of Lemma 3.5 and also Theorems 1.2 and 1.4.  $\square$

Proposition 3.6 below is not used in this paper, but is of independent interest. In Proposition 3.6, we do not assume that the pair  $(X, D)$  is dlt.

**Proposition 3.6.** *Let  $X$  be a normal projective variety of dimension  $n = 1$  or  $2$ , and  $D$  a reduced Weil divisor such that  $K_X + D$  is  $\mathbb{Q}$ -Cartier. Suppose that  $X \setminus D$  is ABH. Then  $K_X + D$  is  $\mathbb{Q}$ -linearly equivalent to an effective divisor, and hence pseudo-effective.*

*Proof.* The case  $n = 1$  is clear. Consider the case of surface  $X$ . Take a dlt blowup (see [2, Theorem 10.4])  $\sigma : X' \rightarrow X$  and let  $E_\sigma$  be the reduced exceptional divisor. We have

$$K_{X'} + B + E = \sigma^*(K_X + D),$$

where the pair  $(X', B)$  is  $\mathbb{Q}$ -factorial dlt with  $B := \sigma^*D + E_\sigma$  a reduced divisor and  $E \geq 0$  a  $\sigma$ -exceptional divisor. Note that  $X' \setminus B$  is an open subset of  $X \setminus D$ , hence hyperbolic, and  $\sigma_*(K_{X'} + B) = K_X + D$ . Thus replacing  $(X, D)$  by  $(X', B)$  we may assume that  $(X, D)$  is already dlt.

Now, as in [8, Theorem 3.4.7], we run the LMMP for  $(X, D)$  and get a composition  $\tau : X \rightarrow X''$  of birational contractions of  $(K_X + D)$ -negative extremal rays such that

$$K_X + D = \tau^*(K_{X''} + D'') + E''$$

with  $D'' := \tau_*D$ ,  $(X'', D'')$  dlt (and hence  $X''$  klt), and  $E'' \geq 0$  a  $\tau$ -exceptional divisor; see [8, Lemma 3.38]. We show that  $K_X + D$  is pseudo-effective so that it is  $\mathbb{Q}$ -linearly equivalent to an effective divisor by the abundance theorem in dimension  $\leq 3$  (see [8, §3.13]).

Suppose the contrary that  $K_X + D$  is not pseudo-effective. Then we may assume that there is an extremal Fano contraction  $X'' \rightarrow Z$  with a general fibre  $G$ . We will reach a contradiction with the hyperbolicity of  $X \setminus D$ .

If  $\dim Z = 1$ , then  $G \cong \mathbb{P}^1$ . Since  $(K_{X''} + D'') \cdot G < 0$ , we have  $D'' \cdot G \leq 1$ . Hence the open subset  $G \setminus D''$  of  $X \setminus D$ , with  $G$  chosen to avoid the set  $\Sigma$  of fundamental points

of  $\tau^{-1}$ , would contain  $\mathbb{C}$ , a contradiction. Thus,  $\dim Z = 0$ . Then the Picard number  $\rho(X'') = 1$  and  $-(K_{X''} + D'')$  is ample.

Case(A).  $D'' = \emptyset$ . Then  $D \subset X$  is contractible to klt singularities on the klt surface  $X''$ . By [5, Theorem 1.3],  $X'' \setminus \Sigma \subseteq X \setminus D$  is dominated by a family of images of  $\mathbb{C}$ , contradicting the hyperbolicity of  $X \setminus D$ . *Note that in this case, the original  $D$  on the original surface  $X$  is also contractible and is a disjoint union of rational trees, and so the pair  $(X, D)$  cannot be hyperbolic.*

Case(B).  $D'' \neq \emptyset$ . By Miyanishi-Tsunoda [10],  $X'' \setminus D''$  either contains an affine-ruling or a platonic  $\mathbb{C}^*$ -fibration, with a general fibre  $F$ . Choose  $F$  to avoid the set  $\Sigma$ . Then  $F \subset X'' \setminus (D'' \cup \Sigma) \subseteq X \setminus D$  contains  $\mathbb{C}^*$ , contradicting the hyperbolicity of  $X \setminus D$ . Instead of [10], one may also apply [5, Theorem 1.3] in this case.  $\square$

**Remark 3.7.** If the pair  $(X, D)$  in Proposition 3.6 is also dlt and  $K_X + D$  is not big, then  $K_{X''} + \tau_* D \sim_{\mathbb{Q}} 0$  in notation of the proof, using the hyperbolicity of a general (curve) fibre minus  $D$ .

## REFERENCES

- [1] A. Corti et al., Flips for 3-folds and 4-folds, Oxford Lecture Ser. Math. Appl. **35**, Oxford Univ. Press, Oxford, 2007.
- [2] O. Fujino, Fundamental theorems for the log minimal model program, Publ. Res. Inst. Math. Sci. **47** (2011), no. 3, 727-789.
- [3] C. D. Hacon and J. McKernan, On Shokurov's rational connectedness conjecture, Duke Math. J. **138** (2007), no. 1, 119-136.
- [4] Y. Kawamata, On the length of an extremal rational curve, Invent. Math. **105** (1991), no. 3, 609-611.
- [5] S. Keel and J. McKernan, Rational curves on quasi-projective surfaces, Mem. Amer. Math. Soc. **140** (1999), no. 669.
- [6] S. Kobayashi, Hyperbolic Manifold and Holomorphic mappings, Marcel Dekker, 1970.
- [7] J. Kollár and 14 authors, Flips and abundance for algebraic threefolds, Astérisque No. **211** (1992).
- [8] J. Kollár and S. Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Math. **134**, Cambridge Univ. Press, 1998.
- [9] S. Lang, Introduction to complex hyperbolic spaces, Springer-Verlag, New York, 1987.
- [10] M. Miyanishi and S. Tsunoda, Logarithmic del Pezzo surfaces of rank one with noncontractible boundaries, Japan. J. Math. (N.S.) **10** (1984), no. 2, 271-319.
- [11] S. Mori and S. Mukai, The uniruledness of the moduli space of curves of genus 11, Algebraic geometry (Tokyo/Kyoto, 1982), 334-353, Lecture Notes in Math., **1016**, Springer, Berlin, 1983.

UNIVERSITY OF QUEBEC AT MONTREAL

*E-mail address:* `lu.steven@uqam.ca`

DEPARTMENT OF MATHEMATICS

NATIONAL UNIVERSITY OF SINGAPORE, 10 LOWER KENT RIDGE ROAD, SINGAPORE 119076

*E-mail address:* `matzdzq@nus.edu.sg`